Solutions 1b

Problem 1: Sum and product notation. Wrtie out the following sums and products and simplify when possible.

1. $\sum_{k=1}^{4}(k-1)^2 = (1-1)^2 + (2-1)^2 + (3-1)^2 + (4-1)^2 = 0 + 1 + 4 + 9 = 14$
2. $\prod_{i=2}^{10} \frac{(i+1)}{i} = \frac{3}{2}\frac{4}{3}\frac{5}{4}\cdots\frac{10}{9}\frac{11}{10} = \frac{11}{2}$
3. $\prod_{j=1}^{3} x^j = x \cdot x^2 \cdot x^3 = x^{1+2+3} = x^6$

Problem 2: Exponents and logarithms.

1. Using the definition of raising a number to a power, explain why $a^3 \cdot a^4 = a^{3+4}$.

   a^3 is equal to multiplying 3 copies of a together, and a^4 is equal to multiplying 4 copies together, so the product of a^3 and a^4 is equal to multiplying $3 + 4 = 7$ copies of a together:

   $$a^3 \cdot a^4 = (a \cdot a \cdot a) \cdot (a \cdot a \cdot a \cdot a) = a^{3+4}.$$

2. Simplify the following:

   (a) $\log(e^2) = 2$

   (b) $\log_2(4 \cdot 16^2) = \log_2(4) + 2\log_2(16) = 2 + 2(4) = 10$

   (c) $y^3 y^{-2} = y^{3-2} = y$

   (d) $z^{\frac{1}{2}} = \sqrt{z}$

Problem 3: Linear equations. Suppose the supply curve for oil is expressed with the following linear equation:

$$-x + 4y = 8$$

And the demand curve is expressed with this equation:

$$2x + 5y = 15$$

1. Find the slope of the supply curve using two points.

   Consider the points $(0, 2)$ and $(-8, 0)$. Then the slope is

   $$\frac{2-0}{0-(-8)} = \frac{1}{4}.$$

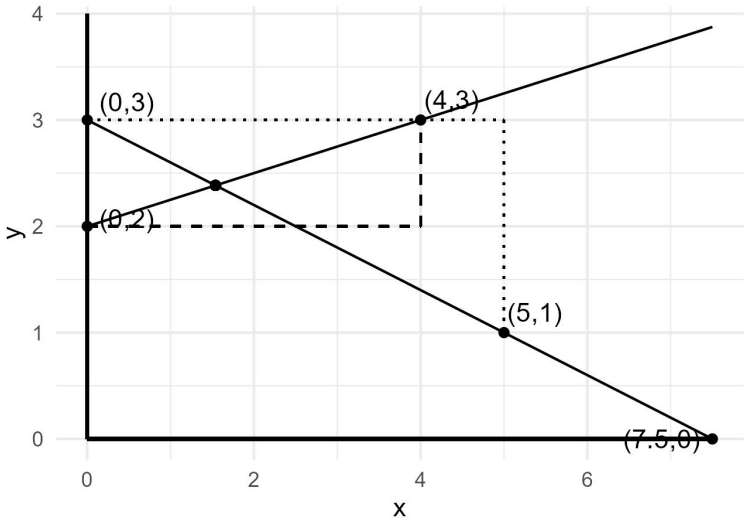
2. Find the slope and intercept of the supply curve and the demand curve by rearranging the equations to be in the form $y = mx + b$.

   The supply and demand curves can be written as

   $$y = \frac{1}{4}x + \frac{8}{4} = \frac{1}{4}x + 2 \text{ and } y = -\frac{2}{5}x + \frac{15}{5} = \frac{2}{5} + 3.$$

   Thus the slopes are $\frac{1}{4}$ and $-\frac{2}{5}$ respectively.

3. Plot the two lines (a simple sketch is fine, be sure to put numbers on the x and y axes). Visually estimate the x and y value of the point where the two lines intersect.



4. Solve for the x and y value where the two lines intersect.

   Above we solved each equation for y. Since we know the left hand side of each equation is y, we can substitute the y in the first equation for the right hand side in the second equation.

$$\begin{aligned}
\frac{-2}{5}x + 3 &= \frac{1}{4}x + 2 \\
\Rightarrow 3 - 2 &= \frac{1}{4}x + \frac{-2}{5}x \\
\Rightarrow 1 &= (\frac{1}{4} + \frac{2}{5})x \\
\Rightarrow 1 &= (\frac{1}{4} \times \frac{5}{5} + \frac{2}{5} \times \frac{4}{4})x \\
\Rightarrow 1 &= (\frac{5}{20} + \frac{8}{20})x \\
\Rightarrow 1 \times \frac{20}{13} &= \frac{13}{20}x \times \frac{20}{13} \\
\Rightarrow \frac{20}{13} &= x.
\end{aligned}$$

   Now, substituting x into either equation:

$$\begin{aligned}
y &= \frac{1}{4} \times \frac{20}{13} + 2 \\
\Rightarrow y = \frac{20}{4 \cdot 13} + 2 &= \frac{1}{13}\frac{20}{4} + 2 \\
\Rightarrow \frac{5}{13} + \frac{26}{13} &= \frac{31}{13}
\end{aligned}$$

Problem 4: Quadratic equations. Consider the equation $x^2 - 8x + 15 = 0$

1. Compute $b^2 - 4ac$. What does this tell you about how many roots the equation has?

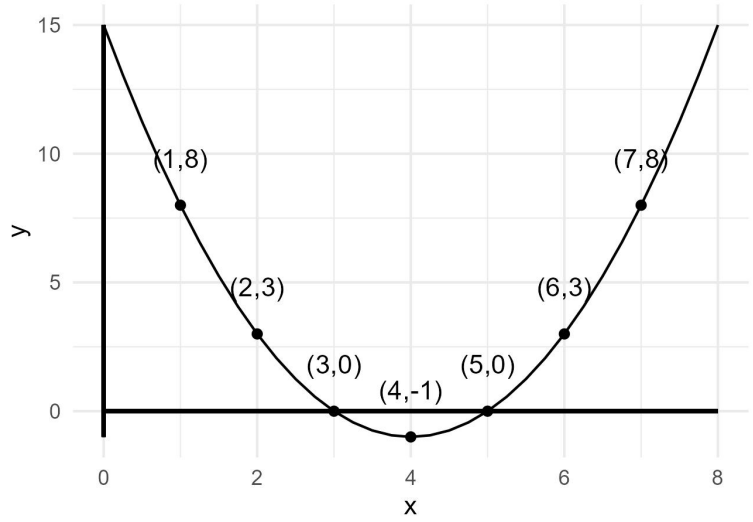
$$b^2 - 4ac = (-8)^2 - 4(1)(15) = 64 - 60 = 4.$$

   Since $b^2 - 4ac > 0$, the equation has two roots (i.e. there are two distinct values of x for which the left hand side is equal to 0).

2. Find the root(s) of the equation using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(15)}}{2(1)} = \frac{8 \pm \sqrt{4}}{2} = 4 \pm 1 = \{3, 5\}.$$

3. Plot the function $f(x) = x^2 - 8x + 15$ by plugging in the x values $1, 2, 3, 4, 5, 6, 7$.



4. What are the domain and range of $f(x)$. Hint: $f(x)$ achieves its minimum y-value at $x = 4$.

   The domain of $f(x)$ is all real numbers since the function is defined for any value of x. The range of $f(x)$ is $[-1, \infty]$ since $f(x)$ can take on values as low as -1 (for $x = 4$) and can take on arbitrarily high values (since $\lim_{x\to\infty} = \infty$).

Problem 5: Limits Limits can be useful for getting a general sense of what a function looks like. Consider the function $g(x) = \frac{1}{x-1}$.

1. Compute the following limits:

   (a) $\lim_{x\to\infty} g(x) = \frac{1}{\infty} = 0$. As x gets very large, the denominator of $g(x)$ is a large number, and 1 diveded by a large number is 0.

   (b) $\lim_{x\to-\infty} g(x) = \frac{1}{-\infty} = 0$. As x gets to be a very large negative number, the denominator of $g(x)$ is a large negative number, and 1 diveded by a large (negative) number is 0.

   (c) $\lim_{x\downarrow 1} g(x) = \infty$. As x approaches 1 from the positive direction, the denominator $x-1$ becomes a very small positive number, and 1 divided by a very small positive number is ∞. To see this, try computing $\frac{1}{0.01}, \frac{1}{0.001}, \frac{1}{0.0001}$, etc.

   (d) $\lim_{x\uparrow 1} g(x) = -\infty$. As x approaches 1 from the negative direction, the denominator $x-1$ becomes a very small negative number, and 1 divided by a very small negative number is $-\infty$.

2. Is there any value of x such that $g(x) = 0$ (i.e. does $g(x)$ cross over the y-axis)?

   If $x > 1$, then the denominator of $g(x)$ is positive, so $g(x) > 0$. If $x < 1$, then the denominator is negative, so $g(x) < 0$. If $x = 1$, then the denominator is 0, so $g(x)$ is

undefined (since $\frac{1}{0}$ is undefined). Thus there is no value of x where $g(x) = 0$ and so the function does not cross over the x axis.

3. Based only on your answers to the above, sketch what the graph of $g(x)$ might look like.