Solutions 2

Problem 1: Matrix multplication

1. Give an example of two matrices A and B such that both AB and BA exist. What are the dimensions of AB and BA?

   In order to have both AB and BA exist, the number of rows in A must be the same as the number of columns in B, and the number of columns in A must be the same as the number of rows in B. For example if A ia a 4×2 matrix (4 rows and 2 columns) and B is a 2×4 matrix (2 rows and 4 columns), then both products exist with AB having dimension 4×4 and BA having dimension 2×2.

2. Give an example of two matrices C and D such that CD exists but DC does *not* exist. What are the dimensions of CD?

   In order for CD to exist, the number of columns in C must be the same as the number of rows in D. In order for DC not to exist, the number of rows in C must be different from the number of columns in D. For example is C is 4×2 and D is 2×3, the CD has dimension 4×3 but DC does not exist.

3. Give an example of a 2×2 matrix with no zero entries whose determinant is 0.

   A 2×2 matrix will have determinant 0 if one column is a scaler multiple of the other, for example:

   $$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

   The determinant of this matrix is $1 \times 6 - 2 \times 3 = 0$.

   More generally, we can represent a matrix whose columns are scaler multiples of each other as

$$\begin{bmatrix} a & ra \\ c & rc \end{bmatrix},$$

where r is any real number other than 0. The determinant of this matrix is $a(rc) - c(ra) = acr - acr = 0$.

4. **[Extra credit]** Describe why $X^T X$ will be a square matrix for any matrix X. Will XX^T also be a square matrix? Can $X^T X$ and XX^T have different dimensions?

   Suppose that the dimensions of X are $n \times m$. Then the dimensions of X^T are $m \times n$. Then the dimension of $X^T X$ is $m \times m$ and the dimension of XX^T is $n \times n$. If $n \neq m$, then $X^T X$ and XX^T have different dimensions.

Problem 2: Solving systems of equations.
Consider the following system of equations:

$$2x + y = 10 \text{ and } x + 4y = 27.$$

1. Represent this system of equations in the form $Az = w$, where A is a 2×2 matrix of numbers, w is a 2×1 matrix of numbers, and z is a 2×1 matrix of variables.

   $$A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, z = \begin{bmatrix} x \\ y \end{bmatrix}, w = \begin{bmatrix} 10 \\ 27 \end{bmatrix}.$$

2. Find $det(A)$. What does this tell you about whether or not A has an inverse?

   $$det(A) = 2(4) - 1(1) = 8 - 1 = 7.$$

   The fact that that $det(A) \neq 0$ tells us that A has an inverse.

3. Find A^{-1} and show that AA^1 is equal to the identity matrix.

   $$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}.$$

Then we can check AA^{-1}:

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \times \frac{1}{7} \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{7} \begin{bmatrix} 2(4) + 1(-1) & 2(-1) + 1(2) \\ 1(4) + 4(-1) & 1(-1) + 4(2) \end{bmatrix} \\ &= \frac{1}{7} \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

4. Multiply both sides of the equation $Az = w$ by A^{-1} to find z.

$$\begin{aligned} Az &= w \\ A^{-1}Az &= A^{-1}w \\ Iz &= A^{-1}w \\ z &= A^{-1}w \\ &= \frac{1}{7} \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 27 \end{bmatrix} \\ &= \frac{1}{7} \begin{bmatrix} 4(10) - 1(27) \\ -1(10) + 2(27) \end{bmatrix} \\ &= \frac{1}{7} \begin{bmatrix} 13 \\ 44 \end{bmatrix} \\ &= \begin{bmatrix} 13/7 \\ 44/7 \end{bmatrix} \end{aligned}$$

5. Check your answer by plugging into the system of equations above.

   Above we found that $x = \frac{13}{7}$ and $y = \frac{44}{7}$. Plugging into the first equation we have

$$2x + y = 2 \cdot \frac{13}{7} + \frac{44}{7} = \frac{26}{7} + \frac{44}{7} = \frac{70}{7} = 10.$$

Plugging into the second equation, we have

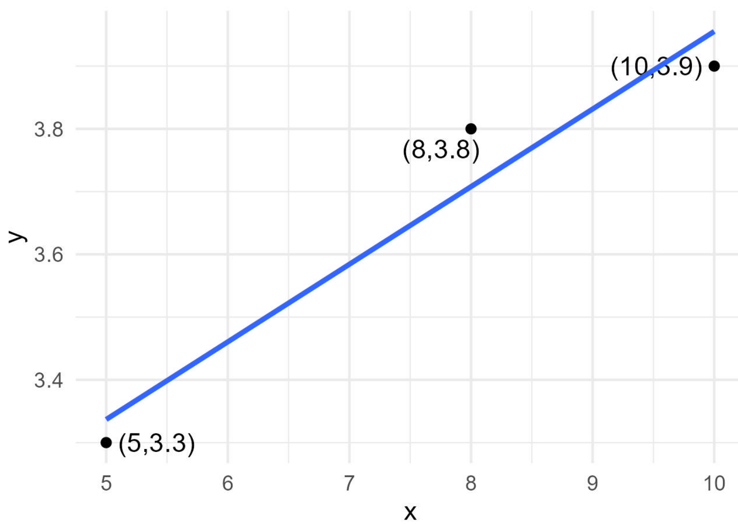
$$x + 4y = \frac{13}{7} + 4 \cdot \frac{44}{7} = \frac{13}{7} + \frac{176}{7} = \frac{189}{7} = 27..$$

Thus the solution $x = \frac{13}{7}$, $y = \frac{44}{7}$ satisfies both equations.

Problem 3: Linear regression. Suppose that you wish to study the association between students' GPA and the number of hours they sleep per night. To study this question, you ask three students to report their GPA and their average hours of sleep. Below is the data:

Student ID	Sleep (hrs)	GPA
1	8	3.8
2	5	3.3
3	10	3.9

1. Plot GPA versus sleep and draw a line that seems to fit the data. Is the slope of this line positive or negative? What does this tell us about the relationship between sleep and GPA?



The best fit line has a positive slope, which suggests that there is a positive association between sleep and GPA. In other words, people who sleep more tend to have higher GPAs (though this does not mean that getting more sleep *causes* higher GPA).

2. Write down the X and y matrices for a linear regression of GPA on sleep. Don't forget to include a colum of 1s in X

$$X = \begin{bmatrix} 1 & 8 \\ 1 & 5 \\ 1 & 10 \end{bmatrix}, y = \begin{bmatrix} 3.8 \\ 3.3 \\ 3.9 \end{bmatrix}.$$

3. Find $X^T X$.

$$\begin{aligned} X^T X &= \begin{bmatrix} 1 & 1 & 1 \\ 8 & 5 & 10 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 1 & 5 \\ 1 & 10 \end{bmatrix} \\ &= \begin{bmatrix} 1(1)+1(1)+1(1) & 1(8)+1(5)+1(10) \\ 1(8)+1(5)+1(10) & 8(8)+5(5)+10(10) \end{bmatrix} \\ &= \begin{bmatrix} 1+1+1 & 8+5+10 \\ 8+5+10 & 64+25+100 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 23 \\ 23 & 189 \end{bmatrix}. \end{aligned}$$

4. Find $(X^T X)^{-1}$.

   The first step to finding the inverse is to compute the determinant:

$$det(X^T X) = 3(189) - 23(23) = 567 - 529 = 38.$$

   Now we can find the inverse:

$$(X^T X)^{-1} = \frac{1}{det(X^T X)} \begin{bmatrix} 189 & -23 \\ -23 & 3 \end{bmatrix} = \frac{1}{38} \begin{bmatrix} 189 & -23 \\ -23 & 3 \end{bmatrix}$$

5. Write down how you would estimate β (you do not need to simplify, simply write down the matrices that you would multiply to find β).

$$\hat{\beta} = (X^T X)^{-1} X^T y = \frac{1}{38} \begin{bmatrix} 189 & -23 \\ -23 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 8 & 5 & 10 \end{bmatrix} \begin{bmatrix} 3.8 \\ 3.3 \\ 3.9 \end{bmatrix}.$$