Practice Problems for Day 3: Differential Calculus

Problem 1: Approximating the derivative. In this problem, we will approximate the derivative of a function at a point by computing the slope of lines between two points. Consider the function $f(x) = x^2$.

1. Use the derivative rules to find $f'(x)$. What is $f'(1)$?

   Using the rule for derivatives of powers, we have $f'(x) = 2x$. Plugging in 1, we have $f'(1) = 2$. This means that the "slope" of the parabola at $x = 1$ is equal to 2.

2. We can approximate the derivative at $x = 1$ as the slope of the line between the points $(1, f(1))$ and $(1 + \delta, f(1 + \delta))$. For $\delta = 1, \frac{1}{2}, \frac{1}{4} . \frac{1}{8}, \frac{1}{16}$, approximate $f'(1)$ as

   $$f'(1) \approx \frac{f(1+\delta) - f(1)}{\delta}.$$

   We can simplify the expression above as

   $$f'(1) \approx \frac{f(1+\delta) - f(1)}{\delta} = \frac{(\delta^2 + 2\delta + 1) - 1}{\delta} = \frac{\delta^2 + 2\delta}{\delta} = \delta + 2.$$

   Then we can compute the following series of approximations:

   $$\begin{aligned} \delta = 1 &\to f'(1) \approx 2 + 1 = 3 \\ \delta = 1/2 &\to f'(1) \approx 2 + 1/2 = 2.5 \\ \delta = 1/4 &\to f'(1) \approx 2 + 1/4 = 2.25 \\ \delta = 1/8 &\to f'(1) \approx 2 + 1/8 = 2.125 \\ \delta = 1/16 &\to f'(1) \approx 2 + 1/16 = 2.0625. \end{aligned}$$

3. What happens to the estimate of $f'(1)$ as δ gets closer to 0?

   As δ gets closer and closer to 0, the approximation for the derivative $f'(1)$ gets closer and closer to its true value of 2.

Problem 2: Calculating derivatives. Compute the following derivatives:

1. $f(x) = x^5 + 2$

   $f'(x) = 5x^{5-1} + 0 = 5x^4$

2. $f(\mu) = 2\mu^4 + \mu^2$

   $f'(\mu) = 2 * 4\mu^{4-1} + 2\mu^{2-1} = 8\mu^3 + 2\mu$

3. $f(y) = ye^y$

   Using the product rule:

$$f'(y) = y\left(\frac{d}{dy}e^y\right) + \left(\frac{d}{dy}y\right)e^y = ye^y + (1)e^y = (y+1)e^y$$

4. $f(x) = \frac{e^x}{e^x+1}$

   Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(e^x+1)\left(\frac{d}{dx}e^x\right) - e^x\left(\frac{d}{dx}(e^x+1)\right)}{(e^x+1)^2} \\ &= \frac{(e^x+1)e^x - e^x(e^x)}{(e^x+1)^2} \\ &= \frac{e^{2x} + e^x - e^{2x}}{(e^x+1)^2} \\ &= \frac{e^x}{(e^x+1)^2} \end{aligned}$$

5. $f(y) = e^{3y^2}$

   We use the chain rule with $g(z) = e^z$ and $h(z) = 3z^2$, Then we have

$$f'(y) = g'(h(y)) \times h'(y) = e^{3y^2}(6y).$$

Problem 3: Finding critical points. Find all critical points of the following functions by setting the derivative equal to 0. For each critical point, find the second derivative and use this to identify the point as a maximum, minimum, or saddle.

1. $f(x) = x^3 + 12$

   $f'(x) = 3x^2 = 0 \to x = 0.$

   $f''(x) = 6x.$ $f''(0) = 0.$

   $f(x)$ has a critical point at $x = 0$. Since the second derivative at $x = 0$ is equal to 0, this is a saddle point.

2. $f(x) = x^3 - 3x$

   $f'(x) = 3x^2 - 3 = 0 \to x^2 = 1 \to x = \pm 1.$

   $f''(x) = 6x.$ $f''(-1) = -6,$ $f''(1) = 6.$

   $f(x)$ has two critical points at $x = -1$ and $x = 1$. Since the second derivative at $x = -1$ is negative, this point is a maximum. Since the second derivative at $x = 1$ is positive, this point is a minimum.

3. $f(x) = x^3 + x$

   $f'(x) = 3x^2 + 1 = 0 \to x^2 = -1/3 \to$ no real solution.

   Since there is no real value of x that makes the derivative equal to 0, the function does not have any critical points.